REVIEW OF THE PHD THESIS OF AIDYN KASSYMOV
"*BASIC FUNCTIONAL AND GEOMETRIC INEQUALITIES FOR THE FRACTIONAL ORDER OPERATORS ON HOMOGENOUS LIE GROUPS*"

1. The relevance of the research topic and its relationship with general scientific and national programs

It is well known that classical analysis inequalities (such as the multidimensional Hardy inequality, Sobolev inequality, Gagliardo Nirenberg inequality, Caffarelli Kohn Nirenberg inequality) on Euclidean space and its manifolds play a fundamental role in the theory (embedding and interpolation) of functional spaces, the analysis of (pseudo)differential operators (on smooth manifolds), and other sections of modern analysis. The dissertation under discussion is devoted to the propagation and generalization of precisely these inequalities (hardy, Sobolev, Gagliardo Nirenberg, Caffarelli Kohn Nirenberg) and a number of others.

The needs of the theory of (nonlinear) differential operators for a number of decades have called for the development and generalization of such inequalities in various directions. In the dissertation, generalizations are made simultaneously in two "directions": with further applications in mind, we consider differential operators of fractional order and homogeneous Lie groups instead of (manifolds) of Euclidean space. Here, this is done by synthesizing and further developing fractional calculus and noncommutative analysis on groups.

The origins of fractional calculus in the concept of a fractional derivative in the Riemann Liouville sense of a function of one real variable, introduced in the nineteenth century. Fractional calculus in Euclidean space received a powerful development in the 60s-80s of the last century, which was summarized in the famous monograph by S. G. Samko A. A. Kilbas O. I. Marichev [123] (in the list of literature of the dissertation).

The first fundamental results of noncommutative (harmonic) analysis are reflected in The (now classical) monograph by E. Stein G. Folland [1].

Recently, there has been a rapid (synthetic) development of fractional calculus and noncommutative analysis on Lie groups and their applications to the analysis of differential operators on groups. A large (and now largely decisive) contribution to this development belongs to Professor M. Ruzhansky and his staff. Here it is appropriate to note that among these actively working employees there is a noticeable (slightly graded) group of young mathematicians from Kazakhstan, including a dissertation student. Intermediate results of this (still ongoing) development are summarized in the monographs of V. Fischer–M. Ruzhansky [3] and M. Ruzhansky–D. Suragan [4]. Thus, the relevance of the topic of the dissertation is not in doubt.

2. Scientific results and their validity

The PhD dissertation consists of an Introduction, four chapters, Appendix and the list of cited literature. In the Introduction (Ch. 1), a brief, but at the same time quite complete in this context, history of the issue and a brief summary of the main results of the dissertation are given. Chapter 2 is of an auxiliary character and contains the basic definitions from the theory of Lie groups, which are necessary in subsequent chapters, as well as the definitions of function spaces and the simplest statements related to them; at the end, definitions of a metric space with measure, a hyperbolic space, and a Cartan Hadamard manifold are given. Chapters 3 to 5 are central to the dissertation. More specifically, in Chapters 3 and 4, the main analytical toolkit is developed, which is then used in Chapter 5 to study differential operators.

In Chapter 3, the PhD candidate prove the main direct inequalities, namely,

- Fractional inequalities of Hardy (Section 3.1), Sobolev (Section 3.2), Hardy - Sobolev (Section 3.3), Gagliardo - Nirenberg (Section 3.4), Caffarelli - Kohn - Nirenberg (Section 3.5);
- the logarithmic inequality of Caffarelli- Kohn - Nirenberg (Section 3.6);
- Hardy Littlewood Sobolev inequality (embedding) for Riesz potentials on homogeneous finite-dimensional Lie groups (Sec. 3.7);
- Stein - Weiss inequality (weighted analogue of the inequality H-L-S from Section 3.7; Section 3.8);
- the logarithmic Sobolev Folland Stein inequality (a logarithmic generalization of the Sobolev Folland Stein inequality from [1]; Sec. 3.9).

In Chapter 4, the PhD candidate show the main "reverse" inequalities (that is, inequalities of the "reverse" sign when the integral parameters p and q $(q < p < 0)$ or $q < 0 < p < 1)$, namely,

- reverse integral Hardy inequality on a metric space with measure (Sections 4.14.2), on a homogeneous Lie group (4.44.4), on a hyperbolic space (Section4.5), and in the Cartan Hadamard manifold;
- reverse inequalities H -L - S and Stein - Weiss on a homogeneous Lie group (Sections 4.7 - 4.10);
- reverse Hardy and Sobolev inequalities with a radial derivative on a homogeneous Lie group (Sections 4.11 - 4.12);
- reverse inequality C - K - N on the homogeneous Lie group (Section 4.13).

In Chapter 5 considers applications of the inequalities obtained in Chapters 3 - 4 to a series of (neo) classical analysis problems in the case of (generally speaking) Lie groups. All problems considered in Chapter 5 are studied according to well-known (classical) schemes developed in the case of (manifolds) of Euclidean space, and for this purpose the corresponding analogues of classical inequalities were established in the two previous chapters.

Thus, in Section 5.1, the PhD candidate prove an analogue of the classical one-dimensional Lyapunov inequality for the Sturm - Liouville operator on a finite segment for a fractional p-sublaplacian on the set (of finite internal quasi-radius) of a Lie group and, as a consequence, the corresponding analogue of the lower bound for the first eigenvalue of this operator. In the case of a set of finite Lebesgue measure in a Euclidean space, this result was previously proved in [69]. Section 5.2 summarizes the results of section 5.1 to the case of systems of fractional p-sublaplacians.

Section 5.3 establishes the existence of a weak solution to the homogeneous Dirichlet problem for a p-sublaplacian with a nonlinear "source" or nonlinear right-hand side on the Heisenberg group (Theorem 5.19) and (as a direct generalization) on the stratified Lie group (Theorem 5.20).

In Section 5.4, the PhD candidate prove the non-uniqueness of a positive weak solution for a sublaplacian with a "Hardy potential" in the case of a domain with a smooth boundary for the Heisenberg group (Theorem 5.26) and (as a direct generalization) of the stratified Lie group (Theorem 5.28). Section 5.5 proves the non-uniqueness of a weak solution of a semilinear equation for a fractional sublaplacian with a "Hardy potential" on a bounded domain with a smooth boundary of a Lie group (Theorem 5.34).

In Sections 5.65.10, the PhD candidate obtained a series of results on the "blow-up" or "non-blow-up" of solutions of the heat equation and the viscoelastic equation on a stratified Lie group, as well as for the heat equation with a fractional sublaplacian and logarithmic nonlinearity on a homogeneous Lie group, the Rockland equation (wave and heat equations) on a graded Lie group. It should be noted here that for the classical equations of mathematical physics and their systems in (areas in) Euclidean space, a powerful theory of "blow-up" of solutions was started in the classical works of H. Fujita of the 1960s. (see, for example, [106]) and, basically, it was completed in the works of S. I. Pohozhaev and his employees of the 1990s 2000s. (for details, see the famous monograph by E. Mitidieri - S.I. Pokhozhaev in Proceedings of the MI RAS, 2001, v. 234).

Finally, in the Appendix in the simplest case (of a segment or) of the real axis, fractional analogues of the classical Hardy, Poincaré, G - N, C - K - N inequalities are obtained, when the definitions of the fractional derivative in the sense of R - L, Hadamard, Caputo are used, and some their applications. In addition, inequalities of the Lyapunov and Hartman-Wintner type are established for fractional partial differential equations with Dirichlet conditions (using the first eigenvalue of the operator to the lower bound) and De La Valleé-Poussin-type inequalities for a fractional elliptic boundary value problem.

3. The degree of validity and reliability of each scientific result, proof and conclusions of the applicant, formulated in the dissertation

All the main results of the dissertation, including those listed above, are new and are in line with modern geometric analysis, are provided with rigorous mathematical proofs, and therefore are completely justified and reliable; as a result, the conclusions and conclusion of the dissertation are justified and reliable.

4. The degree of novelty of each scientific result (position), the conclusion of the applicant, formulated in the dissertation

The scientific results of the dissertation and the list of publications of the thesis on its topic fully meet all the requirements for dissertations for the degree of doctor of philosophy in mathematics.

5. Theoretical and practical significance of the results

The thesis is theoretical, and the results obtained in it can be applied to the further development of the theory of functional inequalities and the analysis of differential operators on geometric structures.

6. Comments and suggestions on the dissertation

In the text of the PhD dissertation, there are some stylistic inaccuracies that need to be corrected. This comment is not significant, it is mainly of a recommended nature and do not reduce the dignity of the dissertation work.

7. The relevant content of the dissertation is within the framework of the requirements of the Rules for awarding academic degrees

The PhD candidate demonstrated a high level of mathematical knowledge and mathematical culture, sufficient for successful independent scientific work at a high mathematical level. Based on the above, I believe that Aidyn Kassymov's dissertation fully deserves to be awarded the degree of doctor of philosophy in mathematics.

Official reviewer,
Professor

D. Bazarkhanov